Algebra and Topology of Lattices of Subgroups

Simplicial complex. ✓ (Russ Woodroofe, 8 Nov 2021.)

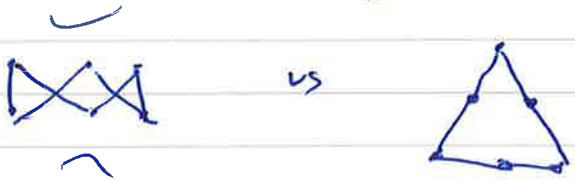
Defn: The order complex of a bounded poset P w/ bottom $\hat{0}$ top $\hat{1}$

- is the simplicial complex
  w/ vertex set $P \setminus \{\hat{0}, \hat{1}\}$
  and faces the chains in $P \setminus \{\hat{0}, \hat{1}\}$.
  (linearly ordered subsets.)

Ex: (Torus)

Ex: Boolean Lattice $B_n = \{$ subsets of $[n]\}$
has $\Delta B_n \cong$ barycentric subdiv.
of bdry of $(n-1)$-simplex.

Eg:



vs

Consider: $L(G)$ the poset of sgps of group G.
and $\Delta L(G)$

Application 1: (Well-known)

Let $\mathrm{Gen}(G,s)$ be # s-tuples $(g_1, \ldots, g_s)$
s.t. $G = \langle g_1, \ldots, g_s \rangle$

Inclusion-exclusion

$$\mathrm{Gen}(G,s) = |G|^s - \sum_{K \text{ max'l sg } G} |K|^s + \sum_{__} \cdot |K_1 \cap K_2|^s$$

$$- \quad - \quad + \quad -$$

$$= \sum_{K \leq G} \tilde{\chi}(\Delta[K,G]) \cdot |K|^s$$

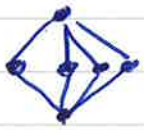
(= Möbius fcn)

$\Delta[K,G]$ = interval between K, G

Application 2:

Determining properties of group G from $L(G)$ or $\Delta L(G)$

Can't determine isom. type or whether G is abelian

$L(S_3) \cong L(\mathbb{Z}_3^2) \cong$ 

Solvability of G can be determined.

(Suzuki / Zappa 1950s)

explicit criterion from Schmidt in 1960s.

Recall: G is solvable if

- $\exists$ series $1 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_n = G$ w/ each $N_i \trianglelefteq G$, each N_i/N_{i-1} abelian.

and/or

- $\exists$ max'l chain $1 = K_0 \triangleleft K_1 \triangleleft \cdots \triangleleft K_m = G$ so each K_i/K_{i-1} cyclic of prime order.

My favorite characterization:

Thm: (essentially Sharcshian 2001)
TFAE for finite gp G:
i) G is solvable
ii) $\Delta L(G)$ is shellable
iii) $\Delta L(G)$ is sequentially Cohen-Macaulay (sCM)

Here:

shellable is a combinatorial / topological condition,

Can order facets in a nice way.
(Tool to verify sCM condition
+ compute other inv.)

sCM (in Topology)
is a vanishing homology condition
on $\Delta[K_1, K_2]$
over each interval between $K_1 \subseteq K_2$.
in $L(G)$.

sCM (in Algebra)
says that a related ring
(the face ring or Stanley-Reisner ring)
has a filtration whose factors
are Cohen-Macaulay.

Interesting parts of pf are (i) $\Rightarrow$ (ii)
and (iii) $\Rightarrow$ (i)

I know 3 pfs of (i) $\Rightarrow$ (ii)

a) Original pf of Shareshian.

b) My proof based on chief series of G
$(N_i \trianglelefteq G)$
N_i/N_{i-1} abelian.

Idea: Label each edge in Hasse diagram
$H \subsetneq K$
w/ associated N_i/N_{i-1}
(least i s.t. $N_i H \geq K$).

Refine w/ structure of $[N_{i-1}, N_i]$
when needed,

Lexicographic order on maxl chains
gives a shelling order.
(EL-Labelling)

c) My proof w/ Schweig
based on composition series.
(similar)
$\rightsquigarrow$ gives a class of "shellable" lattices.

Q: "Book" proof of (iii) $\Rightarrow$ (i) ?

References

Order complexes: A. Björner "Topological methods" in Handbook of Combinatorics (1995).

Application 1: P. Hall "The Eulerian function of a group", Quart J. Math, (1936)
K. Brown "The coset poset and probabilistic zeta function of a finite group", J. Algebra (2000)

Application 2: J. Shareshian "On the shellability of the order complex of the subgroup lattice" Trans. Amer. Math. Soc. (2001).
R. Woodroofe "An EL-labelling of the subgroup lattice", Proc. Amer. Math. Soc. (2008).
J. Schweig and R. Woodroofe "A broad class of shellable lattices" Adv. Math. (2017)